Topology problems

David Glickenstein

August 3, 2006

1 Lecture 1: Topological spaces

1. Give all possible topologies (up to homeomorphism) on a set of three points. Give two collections of open subsets of three points which contain both the set of all points and $\varnothing$ but are not topologies.

2. Show that all open intervals (including unbounded ones) in $\mathbb{R}$ are homeomorphic.

3. If $f : X \to Y$ and $g : Y \to Z$ are continuous, show that the composition $g \circ f : X \to Z$ is continuous.

4. If $Y \subset X$ is given the subspace topology, the inclusion map $\iota : Y \to X$ is continuous. Conclude that if $f : Z \to Y$ is continuous, then if we expand the range and consider $\bar{f} : Z \to X$ (the function is defined the same) then $\bar{f}$ is continuous.

5. Show that the quotient topology on $[0,1]/0 \sim 1$ is homeomorphic to $\left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$ with the subspace topology.

6. Show that the two given definitions of topology generated by a basis are equivalent and are, in fact, topologies.

7. Given two topologies $\mathcal{T}$ and $\mathcal{T}'$ on a space X, if $\mathcal{T} \subset \mathcal{T}'$ then $\mathcal{T}$ is said to be coarser than $\mathcal{T}'$ (because it contains fewer open sets) and $\mathcal{T}'$ is said to be finer than $\mathcal{T}$ (because it contains more open sets). Suppose $\mathcal{B}$ is a basis for $\mathcal{T}$ and $\mathcal{B}'$ is a basis for $\mathcal{T}'$. Show that $\mathcal{T}$ is coarser than $\mathcal{T}'$ if and only if for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

8. Show that the quotient topology is the coarsest topology such that the quotient map is continuous.

9. One can generalize the product topology to an arbitrary Cartesian product $\prod_{j \in J} X_j$ in two ways:

(a) The topology generated by basis sets of the form $\prod_{j\in J} U_j$ where U_j is open in X_j. This is called the *box topology.*

(b) The topology generated by the subbasis $\bigcup_{j\in J} \left\{\pi_j^{-1}(U_j) : U_j \text{ open in } X_j\right\}$, where $\pi_k : \prod_{j\in J} X_j \to X_k$ is the projection map assigning $\pi_k\left((x_j)_{j\in J}\right) = x_k$. This is called the *product topology.*

Show that the product topology is generated by the basis $\prod_{j\in J} U_j$ where U_j is open in X_j and all but finitely many U_j are equal to the entire space X_j. Conclude that the box topology is finer than the product topology. Show that the product topology is the coarsest topology such that the projection maps are all continuous.

10. A function $f : X \to Y$ is *continuous at a point* $x \in X$ if for every neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(U) \subset V$. Explain why for a function $f : \mathbb{R} \to \mathbb{R}$, this is the ε-δ definition of continuity. Show that a function is continuous if and only it is continuous at every point.

11. Show that every metric space is Hausdorff.

12. Given a set X, the *finite complement topology* on X is the topology such that the complement of every finite set is open. Show that this is a topology. Show that if X is finite, then this is the discrete topology. Show that the only closed sets are X, $\varnothing$, and finite sets.

13. Consider the following topology on $\mathbb{R}^n$ called the Zariski topology, used in algebraic geometry. We take as the closed sets the sets

$$F(S) = \{x \in \mathbb{R}^n : f(x) = 0 \forall f \in S\}$$

where S is a set of polynomials in n variables. Show that this is a topology on $\mathbb{R}^n$. Show that any two open sets must intersect, and hence the topology cannot be Hausdorff.

2 Lecture 2: Compactness

1. Prove that the sphere is compact.

2. Show that bounded open intervals in $\mathbb{R}$ are not homeomorphic to bounded closed intervals.

3. Show that the only compact subsets of a discrete space are a finite set of points.

4. Recall that the finite complement topology on X is the topology such that the complement of every finite set is open (see exercises from lecture 1). Show that X and every subspace of X is compact, although only X, $\varnothing$, and finite sets are closed. Why aren't all compact subsets closed?

5. Consider the following space. Let X^n be the set of all lines through the origin in $\mathbb{R}^{n+1}$. One can give this a topology as a quotient by letting $X^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$ where $x \sim \lambda x$ for all $\lambda \in \mathbb{R}\setminus\{0\}$. Show that X is compact and Hausdorff. X^n is called real projective space, or $\mathbb{RP}^n$. One may do the same thing taking instead $\mathbb{C}^{n+1}$ and using the equivalence $x \sim \lambda x$ for all $\lambda \in \mathbb{C}\setminus\{0\}$. This space is called $\mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is compact and Hausdorff.

6. Show that if X is compact and Y is Hausdorff and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

7. A metric space is said to be *sequentially compact* if every sequence has a convergent subsequence whose limit is in the space. Show that a metric space is compact if and only if it is sequentially compact.

8. A metric space is said to be *complete* if every Cauchy sequence converges. It is said to be *totally bounded* if for every $\varepsilon > 0$ it can be covered by finitely many balls of radius ε. Show that a metric space is compact if and only if it is complete and totally bounded.

9. A space is *locally compact* if every point has a compact neighborhood. There is a canonical way to add one point to a locally compact Hausdorff space to get a compact space. If X is locally compact Hausdorff, define $\bar{X} = X \cup \{\infty\}$. The open sets of $\bar{X}$ are the open sets of X together with sets $(X \setminus K) \cup \{\infty\}$ where K is a compact subset of X. Prove that $\bar{X}$ is a compact Hausdorff space. Prove that the natural inclusion $X \to \bar{X}$ is continuous and that the image of X is open in $\bar{X}$. $\bar{X}$ is called the *one-point compactification* of X. Discuss the one-point compactification of $\mathbb{R}^n$.

10. The *Cantor set* is defined as follows. Let $E_1 = [0,1]$. Let $E_2 = [0,1] \setminus (1/3, 2/3)$, $E_3 = E_2 \setminus (1/9, 2/9) \cup (7/9, 8/9)$, and successive E_k defined by deleting the middle third of the open intervals in E_{k-1}. The Cantor set is defined as
$$E = \bigcap_{k=1}^{\infty} E_k.$$
Show that the Cantor set E is compact. Show that the Cantor set is uncountable.

11. One can define a metric structure on the closed subsets of a metric space (X, d) as follows. If F, G are closed subsets of X, then
$$d_H(F, G) = \inf\{\varepsilon : F \subset B_\varepsilon(G) \text{ and } G \subset B_\varepsilon(F)\}$$

where $B_\varepsilon(F) = \{x \in X : d(x, F) < \varepsilon\}$ and $d(x, F) = \inf\{d(x, y) : y \in F\}$. Show that d_H defines a metric. The metric d_H is called the Hausdorff metric. Show that if X is compact, then so is the set of closed subsets of X in the metric topology generated by d_H.

3 Lecture 3: Connectedness

1. Find all of the topologies (up to homeomorphism) on four points which make it connected.
2. Prove that the closure of a connected set is connected.
3. Prove that a path connected space is connected.
4. Show that any infinite set is connected in the finite complement topology. Is $[0, 1]$ path connected in the finite complement topology?
5. Let $K = \{1/n : n = 1, 2, 3, \ldots\}$. Define comb space to be $C = ([0, 1] \times 0) \cup (K \times [0, 1]) \cup (0 \times [0, 1])$ with the subspace topology (C is a subset of $[0, 1] \times [0, 1]$). Define the deleted comb space to be $C' = C \setminus 0 \times (0, 1)$. Show that the deleted comb space is connected but not path connected.
6. A space is *totally disconnected* if the only connected subsets are one-point sets. Show that the set of rational numbers between 0 and 1 (i.e. $\mathbb{Q} \cap [0, 1]$) with the subspace topology is totally disconnected. Show that the Cantor set is totally disconnected.
7. Show that none of $(0, 1)$, $[0, 1)$, and $[0, 1]$ are homeomorphic (use the notion of a connected set).
8. Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic (use the notion of a connected set).
9. A space X is said to be *locally connected at x* if for every neighborhood U of x there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points X is said to be *locally connected*. Show that the subset $\{0\} \cup \bigcup_{n=1}^{\infty} \{1/n\}$ is not locally connected. Show that $[-1, 0) \cup (0, 1]$ is locally connected but not connected. Show that the deleted comb space is connected but not locally connected.
10. A space X is said to be *locally path connected at x* if for every neighborhood U of x there is a path connected neighborhood V of x contained in U. If X is locally path connected at each of its points X is said to be *locally path connected*. Show that if X is locally path connected and connected, then it is path connected.
11. Prove that an arbitrary product of connected sets with the product topology is connected. Does your proof hold if the box topology is used?