ALGEBRA QUALIFYING EXAM
FALL 2010

- Do any one of the problems nA or nB where $n = 1, 2, 3, 4, 5$.
- You may use a separate sheet for scratch work.
- Be precise, concise and to the point.
- Each problem is worth 25 points.
- Show all steps and details; say what you mean, mean what you say.

1A: Find the Jordan canonical form of the real matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ a & 3 & 0 \\ b & 0 & -2 \end{bmatrix}$$

Does the Jordan canonical form depend on a, b?

1B: Let F be a field, V a finite dimensional F-vector space and V^* the dual F-vector space. For an F-subspace W in V define the annihilator $\mathrm{Ann}(W) := \{\lambda \in V^* | \lambda(w) = 0 \quad \text{for all} \quad w \in W\}$. Show that the following statement holds: If U, W are F-subspaces of V with $U \subseteq W$ then $\mathrm{Ann}(U)/\mathrm{Ann}(W)$ and $(W/U)^*$ are isomorphic F-vector spaces. Furthermore, give an explicit isomorphism.

2A: Let G be a group with 12 elements. Prove that if the center of G contains no element of order 2 then $G \cong A_4$.

2B: Let G be a finite nonabelian simple group. Show that the order of G can not be of the form $2^m p^n$, where p is a prime, $m = 1, 2, 3$ and $n \in \mathbb{N}$.

3A: Let R be a unique factorization domain. Show:

a) Any non-zero $x \in R$ is contained in finitely many principal ideals.

b) Any ascending chain of principal ideals in R terminates.

3B: Let k be a field. Prove that the ideal generated by the polynomial $X^2 - YZ$ is prime in $k[X, Y, Z]$.

4A: Let F_2 be the field with 2 elements and let $f, g \in F_2[x]$ be the following polynomials $f(x) = x^3 + x + 1$ and $g(x) = x^3 + x^2 + 1$. Construct a splitting field E for f and a splitting field L for g and give an explicit isomorphism from E to L.

4B: Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic field $\mathbb{Q}(\zeta_n)$ (where ζ_n is a primitive n-th root of unity).

5A: Let R be a ring and M be an R-module. Show that M is the (internal) direct sum of two R-submodules N and P with $N \neq M$ and $P \neq M$ if and only if there is an R-endomorphism ϕ of M, not equal to the zero map and not equal to the identity map, such that $\phi^2 = \phi$.

5B: Find all the possible rational canonical forms of a 3×3 real matrix A that satisfies $A^6 = I$.