Algebra Qualifying Examination

January 2016

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. Let A, B be two square matrices over a field F. Suppose that $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ is similar to $\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$. Prove that A is similar to B.

1B. Let F be an infinite field and let V be a F-vector space. Show that if $V = \cup_{i=1}^{n} V_i$ for F-subspaces, $V_1, \ldots, V_n$, then there is $j \in \{1, \ldots, n\}$ with $V = V_j$.

2A. Show that the automorphism group of the quaternion group of order 8 is a semidirect product of a group of order 4 and a group of order 6.

2B. Let G be a finite simple (abelian or non-abelian) group of order n. Find the number of normal subgroups of $G \times G$.

3A. Give a complete proof of the Hilbert Basis Theorem: *If R is a commutative Noetherian ring with identity, then so is $R[x_1, x_2, \ldots, x_n]$.*

3B. Let R be a PID and I a nonzero ideal of R. Show that there are only finitely many ideals of R containing I. Show by example that this may not hold if R is a UFD but not a PID.

4A. Let $F = \mathbb{C}$, let $K = \mathbb{C}(t)$, the field of rational functions in an indeterminate t, and let G be the Galois group $G(K/F)$. Suppose φ and θ in G are determined by $\varphi(t) = \zeta t$ and $\theta(t) = 1/t$, where ζ is a primitive nth root of unity in $\mathbb{C}$, $n \geq 4$, and set $H = \langle \varphi, \theta \rangle \leq G$. Show that H is isomorphic with the dihedral group D of order $2n$ and show that the fixed field of H is $\mathbb{C}(t^n + t^{-n})$.

4B. Let $\xi \in \mathbb{C}$ be such that $\xi^{2015} = 3$. Show that -3 is not a sum of squares in $\mathbb{Q}(\xi)$.

5A. Let G be the group given by the presentation $\langle x, y, z | x^2, y^3, (xyz)^4 \rangle$. Write the commutator factor group $G/[G,G]$ as a direct product of cyclic groups and justify your answer.

5B. Let A be a finite dimensional, associative (not necessarily commutative) $\mathbb{C}$-algebra with no zero divisors, but with identity. Show that $\dim_{\mathbb{C}} A = 1$.