ALGEBRA QUALIFYING EXAMINATION

JANUARY 2021

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. Let V be the space of polynomials of degree at most n. Define a linear operator on V by

$$D : V \to V, \quad D(p(x)) = p'(x) + 2p''(x).$$

Find the characteristic polynomial of D.

1B. Let A be an $n \times n$, real, skew-symmetric matrix. Prove that $\det(I_n + A) \geq 1$.

2A. Let G be a finite group and H, K be two subgroups. Show that for any $g \in G$ we have

$$\#HgK = \#H \cdot [K : g^{-1}Hg \cap K].$$

2B. Classify (up to isomorphism) all finite groups with exactly 3 conjugacy classes.

3A. Show that the subring of $\mathbb{Q}(x)$ given by

$$\left\{ \frac{f(x)}{g(x)} \mid f(x), g(x) \in \mathbb{Q}[x],\ g(0) \neq 0 \right\}$$

has only one maximal ideal. Explicitly describe this maximal ideal.

3B. Let R be a commutative (unital) ring and I, J ideals of R with $I + J = R$. Prove that for all positive integers m, n one has $I^m + J^n = R$.

4A. Let $f(x) \in \mathbb{Q}[x]$ be a degree n polynomial and E be the splitting field of $f(x)$. Show that $[E : \mathbb{Q}]$ divides $n!$.

4B. Let $p_1 < p_2 < \cdots < p_n$ be positive prime numbers, and let K be the extension of $\mathbb{Q}$ obtained by adjoining all $\sqrt{p_i}$ for $1 \leq i \leq n$. Prove that every subfield of K of degree 2 over $\mathbb{Q}$ is of the form $\mathbb{Q}(\sqrt{d})$ with d a product of the elements in some nonempty subset of $\{p_1, p_2, \ldots, p_n\}$.

5A. Let R be a commutative (unital) ring and M a noetherian R-module. Let $\varphi : M \to M$ be an R-module homomorphism. Show the following.

(1) The chain of submodules

$$\operatorname{Ker}\varphi \subset \operatorname{Ker}\varphi^2 \subset \operatorname{Ker}\varphi^3 \subset \cdots$$

stabilizes.

(2) We have

$$\operatorname{Ker}\varphi^n \cap \operatorname{Im}\varphi^n = \{0\}$$

if n is sufficiently large.

(3) If φ is surjective then it is an isomorphism.

5B. Let $M = \mathbb{Z}^3$ and let N be the $\mathbb{Z}$-submodule of M generated by $(2, -1, 0)$, $(3, 4, 1)$, and $(5, -3, 2)$. Express M/N as a direct sum of cyclic $\mathbb{Z}$-modules.