ANALYSIS QUALIFYING EXAMINATION

August 2009

1. Let $f_1, f_2, \dots f_{2^n}$ be 2^n complex-valued functions on $\mathbf{R}$, integrable in the power 2^n: $\int_{\mathbf{R}} |f_j(x)|^{2^n}\, dx < \infty$ for $j = 1, 2, \dots 2^n$. Prove that

$$\int_{\mathbf{R}} |\prod_{j=1}^{2^n} f_j(x)|\, dx \le \prod_{j=1}^{2^n} (\int_{\mathbf{R}} |f_j(x)|^{2^n}\, dx)^{2^{-n}}.$$

2. Let f be a real-valued, integrable function on $[0, 1]$. Calculate

$$\lim_{n\to\infty} \int_0^1 \frac{1}{n} \log(1 + e^{nf(x)})\, dx.$$

Justify your answer.

3. Let μ be a (positive) Borel measure on $\mathbf{R}$, with $\mu[\mathbf{R}] = 1$. Use the Fubini theorem to prove that for every Borel set E

$$\int_{\mathbf{R}} \mu[x + E]\, dx = m[E],$$

where m denotes the Lebesgue measure on $\mathbf{R}$.

4. Use the Parseval identity for the function $x \mapsto e^{iAx}$ with an appropriate A, to prove the formula:

$$\sin^{-2} t = \sum_{n=-\infty}^{+\infty} \frac{1}{(t - n\pi)^2}$$

for real t, $t \ne k\pi$ with integer k.

5. Prove divergence of the numerical series

$$\sum_{n=2}^{\infty} (\log n)^{-\log\log n}.$$

6. Let T be the operator on the Hilbert space $\mathcal{H} = L^2([0, 1])$, defined by:

$$Tf(x) = xf(x).$$

Prove that T is a bounded linear operator on $\mathcal{H}$ and find its operator norm $||T||$.

Please, show all your work!

GOOD LUCK!