Analysis qualifying exam - August 2013

(1) Find

$$\lim_{n\to\infty} n^2 \int_0^\infty e^{-nx} \arctan(x)\,dx$$

and prove your answer. ($\arctan(x)$ is the inverse function for $\tan(x)$.)

(2) Let f_n be a sequence of continuous functions on $\mathbb{R}$ which converge uniformly on $\mathbb{R}$ to a continuous function f. Define

$$g_n(x) = \int_0^x f_n(t)\,dt, \quad g(x) = \int_0^x f(t)\,dt$$

For each of these statements, either prove it or disprove it by a counterexample.

(a) g_n converges to g point-wise.

(b) g_n converges to g uniformly on $\mathbb{R}$.

(c) For all $a < b$, g_n converges to g uniformly on $[a, b]$.

(3) Let $AC([a, b])$ be the set of absolutely continuous functions on the bounded interval $[a, b]$, and define

$$K = \{f : f \in AC([a, b]),\ ||f||_2 \leq 1,\ ||f'||_2 \leq 1\}$$

Prove that the closure of K is compact in $C([a, b])$, the space of continuous functions on $[a, b]$ with the sup norm.

(4) Prove that the linear operator A defined by the formula

$$Au(x) = \int_0^1 \frac{u(t)}{\sqrt{|x-t|}}\,dt$$

is bounded from $L^2([0, 1])$ to $L^2([0, 1])$. (The measure for $L^2([0, 1])$ is Lebesgue measure.) Hint: $|x - t|^{-1/2} = |x - t|^{-1/4}\,|x - t|^{-1/4}$.

(5) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let f and g be non-negative functions in $L^1(X, \mathcal{M}, \mu)$. Define two new measures by

$$\mu_f(A) = \int_A f \, d\mu, \quad \mu_g(A) = \int_A g \, d\mu$$

for $A \in \mathcal{M}$.

(a) Find a necessary and sufficient condition on f and g for μ_f to be absolutely continuous with respect to μ_g.

(b) When μ_f is absolutely continuous with respect to μ_g, what is the Radon-Nikodym derivative of μ_f with respect to μ_g ?

(6) Let λ_n be Lebesgue measure on $\mathbb{R}^n$, and let λ_1 be Lebesgue measure on $\mathbb{R}$. Let $f, g \in L^2(\mathbb{R}^n, \lambda_n)$, and assume $g \geq 0$. For $t \geq 0$ define $E_t = \{x \in \mathbb{R}^n : g(x) > t\}$. Prove that

$$\int_0^\infty \left[\int_{E_t} f(x) \, d\lambda_n(x) \right] d\lambda_1(t) = \int_{\mathbb{R}^n} f(x) g(x) \, d\lambda_n(x)$$