Analysis Qualifying Exam

PLEASE SHOW ALL YOUR WORK

1. Let (X, ρ) be a metric space, $E \subset X$, and $f(x) = \inf_{y \in E} \rho(x, y)$. Show that f is continuous on X, and that $\bar{E} = \{x \in X : f(x) = 0\}$.

2. Suppose $f : [0, \infty) \to \mathbb{R}$ is a Lebesgue integrable function with compact support. Define the Laplace transform of f by
$$F(t) = \int_0^\infty f(x) e^{-tx} dx.$$
Show that F is continuously differentiable on $(0, \infty)$.

3. Justify the statement that
$$\int_0^1 \int_0^1 \frac{(x-y)\sin(xy)}{x^2+y^2} dxdy \;=\; \int_0^1 \int_0^1 \frac{(x-y)\sin(xy)}{x^2+y^2} dydx.$$

4. Let $\{f_n\}_{n \geq 1}$ be a sequence of functions in $L^p(\mathbb{R}, \mu)$ where $1 < p < \infty$ and μ is a Borel measure on $\mathbb{R}$. Suppose that
$$\sup_{n \geq 1} \|f_n\|_p < \infty.$$
Show that $\{f_n\}_{n \geq 1}$ is 'uniformly integrable', that is, for all $\epsilon > 0$ there is a $\delta > 0$ such that when $\mu(E) < \delta$, we have
$$\sup_{n \geq 1} \int_E |f_n| d\mu < \epsilon.$$

5. Let μ and ν be finite non-zero positive measures on the measure space (X, M) such that $\nu << \mu << \nu$. Let $\frac{d\nu}{d(\mu+\nu)}$ represent the Radon-Nikodym derivative of ν with respect to $\mu + \nu$. Show the following strict inequality:
$$0 < \frac{d\nu}{d(\mu+\nu)} < 1 \text{ a.e. with respect to } \mu.$$

6. For $p > 1$, let $f, g \in L^p(\mathbb{R})$. Suppose that $T(f) = T(g)$ for all bounded linear functionals T on $L^p(\mathbb{R})$. Show that $f = g$ in $L^p(\mathbb{R})$.