Geometry-Topology Qualifying Exam

Fall 2014

Problem 1

For $p(z)$ a polynomial of degree n with its zeros lying within distance R from the origin compute

$$\int_{|z|=R} \frac{p'(z)}{p(z)} dz.$$

Be sure to justify your calculation and cite the theorems you are using.

Problem 2

a) For a vector field V and a differential form ω on a manifold M the Lie derivative of ω along V is defined by

$$\mathcal{L}_V \omega := \iota_V d\omega + d(\iota_V \omega).$$

Prove that it is a derivation acting on the ring of differential forms, i.e. that for any pair of differential forms, a p-form η and a q-form τ, we have

$$\mathcal{L}_V(\eta \wedge \tau) = \mathcal{L}_V(\eta) \wedge \tau + \eta \wedge \mathcal{L}_V(\tau).$$

Note: You can make use of the basic properties of exterior derivative and interior and exterior product, once you state them.

b) For the vector field $V = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ and the one-form $\omega = x^2 dy + y^2 dx$ on $\mathbb{R}^2$ compute the Lie derivative $\mathcal{L}_V \omega$.

Problem 3

a) Prove that the group $SL(2, \mathbb{R})$ of all 2×2 real matrices of determinant one,

$$SL(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| ad - bc = 1; a, b, c, d \in \mathbb{R} \right\},$$

is a three-dimensional manifold.

b) Give an example of three global vector fields on $SL(2, \mathbb{R})$ that form a basis for the tangent space at the point $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{R})$.

Problem 4

Let ω be a closed two-form on a compact connected orientable manifold M^{2n} (without a boundary) of dimension $2n$. Assume that ω is a nowhere degenerate form, i.e. for any point p and any bases $e_1, e_2, \ldots, e_{2n}$ in the tangent space T_pM^{2n} the expression

$$\sum_{i_1 i_2 \ldots i_{2n}} \epsilon^{i_1,i_2,\ldots,i_{2n}} \omega(e_{i_1}, e_{i_2}) \omega(e_{i_3}, e_{i_4}) \ldots \omega(e_{i_{2n-1}}, e_{i_{2n}}) \neq 0.$$

Here $\epsilon^{i_1,i_2,\ldots,i_{2n}}$ is a completely antisymmetric tensor with $\epsilon^{1,2,\ldots,2n} = 1$; in other words, $\epsilon^{i_1,i_2,\ldots,i_{2n}}$ is the sign of the permutation $(i_1, i_2, \ldots, i_{2n})$.

Show that the n-th exterior power

$$\omega \wedge \omega \wedge \ldots \wedge \omega$$

generates the top de Rham cohomology group $H^{2n}_{\text{dR}}(M^{2n})$.

Problem 5

The space X is obtained by attaching a Möbius strip along its boundary to one of the meridians of a two-torus (marked γ in the picture below). Find the fundamental group of X for some choice of a base point.

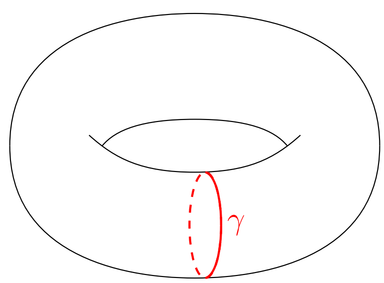

Figure 1: A torus with a marked meridian.

Problem 6

Consider a surface obtained by identifying edges of a square as indicated in Figure 2.

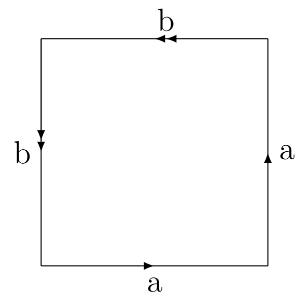

Figure 2: The CW complex of the surface.

a) Directly compute the cellular homology of this space.

b) Give this space a Δ-complex structure and use it to compute its simplicial homology.

c) Let point p be at the center of the above square and D a small open two-disk centered at p. Compute the homology of the surface via the Mayer-Vietoris sequence, using D and the complement of p as the two open sets.