Topology-Geometry Qualifying Exam: Jan 2011

1. Suppose that Ω is an open subset of the complex plane $\boldsymbol{C}$ and $f\colon \Omega \to \boldsymbol{C}$ is a holomorphic map. Identify $\boldsymbol{C}$ with $\boldsymbol{R}^2$ in the usual fashion and let F denote the map f thought of as a map from an open subset Ω of $\boldsymbol{R}^2$ into $\boldsymbol{R}^2$. Let DF_x denote the derivative of F at $x = (x_1, x_2)$. Show that,

$$\det(DF_x) = |f'(z)|^2, \text{where } z = x_1 + i x_2.$$

2. Let $\mathcal{H}$ be the two dimensional submanifold of $\boldsymbol{R}^3$ defined by,

$$p_0^2 - p_1^2 - p_2^2 = 1, \text{with } p_0 > 0.$$

where (p_0, p_1, p_2) are the usual coordinates in $\boldsymbol{R}^3$. Suppose that $u, v \in T_p\mathcal{H}$ (the tangent space to $\mathcal{H}$ at p thought of as a subspace of $T_p\boldsymbol{R}^3$) are given by,

$$u = \sum_{j=0}^{2} u_j \frac{\partial}{\partial p_j}, \text{and } v = \sum_{j=0}^{2} v_j \frac{\partial}{\partial p_j}$$

Define a bilinear form in the tangent space at p to $\mathcal{H}$ by,

$$\langle u, v\rangle = u_1 v_1 + u_2 v_2 - u_0 v_0.$$

Calculate this bilinear form in the coordinates $(x_1, x_2) \in D$ (where D is the open unit disk $x_1^2 + x_2^2 < 1$) for $\mathcal{H}$ given by stereographic projection,

$$p \to x = \left(\frac{p_1}{1 + p_0}, \frac{p_2}{1 + p_0}\right),$$

with inverse,

$$D \ni x \to \left(\frac{1 + |x|^2}{1 - |x|^2}, \frac{2x_1}{1 - |x|^2}, \frac{2x_2}{1 - |x|^2}\right), \text{ with } |x|^2 = x_1^2 + x_2^2.$$

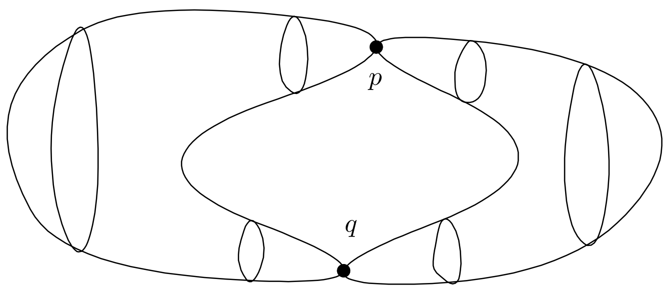
That is, calculate,

$$\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right\rangle, i, j = 1, 2.$$

Use this to show that the bilinear form is an inner product, i.e., $\langle u, u\rangle > 0$ if $u \neq 0$.

3. Use Mayer-Vietoris to calculate the Homology with coefficients in $\boldsymbol{Z}$ of the two torus "pinched" at two points p and q. Draw a picture of the representative cycles.

X



4. Let $\pi\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2/(\boldsymbol{Z}\times\boldsymbol{Z}) \simeq S^1 \times S^1$ be the natural projection on the quotient space. Define,

$$p = \pi(1/2, 1/2) \text{ and } q = \pi(0, 0).$$

Use the Siefert-Van Kampen theorem to determine $\pi_1(S^1 \times S^1 \backslash \{p\}, q)$, the fundamental group of the torus punctured at p with base point q.

Do just 2 of the following 3 problems,

5. Suppose that $p(z)$ is a polynomial of degree n with complex coefficients. Suppose that none of the roots of $p(z)$ has absolute value 1 and that exactly k of the roots of $p(z)$ have absolute value strictly less than 1. Show that the map,

$$S^1 \ni z \to \frac{p(z)}{|p(z)|} \in S^1,$$

is homotopic to the map,

$$S^1 \ni z \to z^k \in S^1,$$

and hence has topological degree k.

6. Let S^n be the n sphere and

$$\pi: S^n \to \boldsymbol{R}P^n$$

the projection on real projective n space. For $p \in S^n$ define the antipodal map ϕ,

$$\phi(p) = -p \in S^n.$$

(a) Show that if $\omega \in \Omega^k(S^n)$ (a k form on S^n) and $\phi^*\omega = \omega$ then there exists a k form ω' on $\boldsymbol{R}P^n$ so that,

$$\omega = \pi^*\omega'$$

(b) Let $\omega \in \Omega^n(S^n)$ be the volume form on S^n defined by,

$$\omega_p(v_1, v_2..., v_n) = dp_0 \wedge dp_1 \wedge \cdots \wedge dp_n(n_p, v_1, ..., v_n),$$

where $dp_0 \wedge dp_1 \wedge \cdots \wedge dp_n$ is the standard volume form on $\boldsymbol{R}^{n+1}$, $n_p = p$ is the unit normal field on S^n and the v_j are all tangent vectors to S^n at $p \in S^n$. Calculate $\phi^*\omega$ and use this to show that $\boldsymbol{R}P^n$ is orientable when n is odd.

7. The configuration of a rigid rod in $\boldsymbol{R}^2$ is determined by the location of the two end points p and q of the rod (which we assume are distinguishable). If the rigid rod has length ℓ then we define,

$$X = \{(p, q) \in \boldsymbol{R}^2 \times \boldsymbol{R}^2 | \, |p - q| = \ell\}.$$

Show that X is a manifold diffeomorphic to $\boldsymbol{R}^2 \times S^1$. Find a one form on X that generates the deRham cohomology $H^1(X)$.