GEOMETRY-TOPOLOGY QUALIFYING EXAM, JANUARY 2017

1. Suppose that $f(z)$ is an analytic function defined in a neighborhood of $z = 0$ with $f(0) = 0$. Suppose that for some sufficiently small neighborhood of $z = 0$ the map $z \to f(z)$ is injective. Is it true that $\frac{f(z)}{z}$ must be analytic and non-zero near $z = 0$? Give a proof or a counter-example.

2. Let

$$M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 4y^2 + 9z^2 = 1\}.$$

(a) Briefly explain why M is an embedded submanifold of $\mathbb{R}^3$.

(b) Equip M with the orientation induced by the outward pointing normal. Evaluate

$$\int_M y dx \wedge dz$$

3. Consider the vector field $X = (4y^3 + x^2 y)\frac{\partial}{\partial x} - (x^3 + xy^2)\frac{\partial}{\partial y}$ in $\mathbb{R}^2$.

(a) Show that $L_X H (= X(H)) = 0$, where $H = y^4 + \frac{1}{4}x^4 + \frac{1}{2}x^2 y^2$.

(b) Use (a) to show that the solution to

$$\frac{dx}{dt} = 4y^3 + x^2 y$$
$$\frac{dy}{dt} = -x^3 - xy^2$$

$x(0) = 5, y(0) = 7$ exists for all time t.

4. Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$. Consider a map $f : S^n \to S^m$ satisfying $f(-x) = -f(x)$ (for $m, n \geq 1$). f gives rise to a map

$$\bar{f} : \mathbb{RP}^n \to \mathbb{RP}^m.$$

Prove that the induced map on fundamental groups

$$\bar{f}_* : \pi_1(\mathbb{RP}^n, x) \to \pi_1(\mathbb{RP}^m, \bar{f}(x))$$

is nonzero.

5. Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Fix positive integers $m < p$, and assume that p is prime. Identify $\mathbb{Z}/p\mathbb{Z}$ with the pth roots of 1, i.e. $\{\zeta \in \mathbb{C} : \zeta^p = 1\}$, and define a group action by

$$\mathbb{Z}/p\mathbb{Z} \times S^3 \to S^3 : (\zeta, (z_1, z_2)) \mapsto (\zeta z_1, \zeta^m z_2)$$

(a) Briefly explain why the quotient M of S^3 by this group action is a manifold.

(b) Compute (up to isomorphism, you do not need to find representatives) $\pi_1(M)$, $H_1(M, \mathbb{Z})$, $H_1(M, \mathbb{R})$, and $H^1_{DR}(M, \mathbb{R})$.

6. Give examples of the following:

(a) A topological space which is connected but not path connected.

(b) A double covering of a figure eight.

(c) A covering space of a figure eight which has $\mathbb{Z} \times \mathbb{Z}$ as its group of automorphisms.